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Moving-mirror entropy

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We consider a quantized scalar field in a two-dimensional Minkowski spacetime with a moving mirror and propose a definition of moving-mirror entropy associated with temporarily inaccessible information about the future.

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One of the most embarrassing problems in gravitational physics is the so-called information loss problem. When Hawking discovered a thermal radiation from black holes (Hawking radiation) in 1974 by using semiclassical and quasi-stationary approximations, he argued that formation and evaporation of black holes would introduce a non-unitarity in quantum mechanics [1]. Actually, his calculation shows that an initial pure state evolves to a totally uncorrelated thermal state, which is impossible in a unitarily evolving system. In general within these approximations, the evolution is described by a non-unitary superscattering matrix which maps initial mixed states to final mixed states, and superscattering matrix elements can be calculated explicitly [2,3]. Hence, as far as the approximations are justified, some information with respect to the initial state seems to be lost.

On the other hand, Page [4] argued that, if the superscattering matrix describing the whole system including quantum gravity is CPT invariant, the description can be reduced to an S matrix which maps pure initial states into pure final states. In other words, if quantum gravity is CPT invariant then there is no loss of information in the process of formation and evaporation of black holes. In this case information can be lost only temporarily and the temporarily missing information should be completely recovered. Although his arguments might seem to contradict the semiclassical result that emission from a black hole is uncorrelated, he also argued later that information may come out initially so slowly, or else be so spread out, that it would never show up in a perturbative analysis [5]. Hence, it seems very difficult to prove or disprove his arguments by any perturbative analysis, in particular the semiclassical calculation.

Page's arguments seem to be consistent with the interpretation of entanglement entropy proposed in [6,7]. Entanglement entropy of a pure state with respect to a division of a Hilbert space into two subspaces 1 and 2 was interpreted as an amount of information which can be transmitted through 1 and 2 from a system interacting with 1 to another system interacting with 2. In this interpretation, the transmitting medium is the quantum entanglement between 1 and 2, and the entanglement entropy may be a quantity which can in principle cancel the black-hole entropy, the amount of the temporarily missing information, to restore information loss. However, so far we do not have any concrete models to realize this idea.

Recent progress in string theory suggests the existence of a unitary S-matrix description for the Hawking process. (For reviews of black holes in superstring theory see, for example, [8–10] and references therein.) In the D-brane picture, an extremal or near extremal black hole is described by stacked D-branes on which open strings are attached and moving. Hawking radiation can be described as a decay process of two open strings into a closed string away from the D-branes. Since the evolution of the whole system in the D-brane picture including closed and open strings is manifestly unitary, the Hawking process should be described as a unitary process at least for near extremal black holes. Moreover, the AdS/CFT correspondence [11,12] might be considered as a conjecture about a unitary description for arbitrary gravitational processes including formation and evaporation of black holes.

The D-brane picture also provides us an interpretation of black-hole entropy of an extremal or near extremal black hole as the logarithm of the number of different states of open strings on stacked D-branes [13]. Hence, in some sense the black-hole entropy seems to be an amount of temporarily inaccessible information. Namely, if the black hole evaporates completely then information about the quantum state of the black hole should come out as correlations among closed strings, provided that a unitary description similar to the D-brane picture holds for evaporating black holes.

However, it seems fair to say that the above consideration about the restoration of missing information is no more than speculation. For example, see Hawking's objections [14]. Moreover, because of Page's arguments that the recovery of information would never show up in a perturbative analysis, it seems very difficult to prove or disprove it.

On the other hand, one might consider a different approach: it might be relevant to investigate whether it is possible to assign entropy to a manifestly unitary process. In this paper, we shall consider the so-called moving-mirror effect [15,16], which is manifestly unitary but is often referred to as an analogue of the Hawking process.

For simplicity, consider a minimally coupled scalar field in two-dimensional Minkowski spacetime ($ds^2=-dt^2+dx^2=-du dv$) with a moving mirror. Denote the trajectory of the mirror by $x^\mu=X^\mu(\tau)$, where τ is the proper time along the mirror trajectory and assume that both sides of the mirror are perfectly reflecting. For this situation, suppose that the functions $X^\mu(\tau)$ for $\tau<\tau_0$ are known and that the remaining $\tau>\tau_0$ part of the trajectory is unknown. Of course, provided that the initial state at $\mathcal{I}^-$ is given, we can determine the quantum state of wave-packet modes which are reflected by the mirror before τ_0. However, we cannot predict the quantum state of wave-packet modes which will be reflected by the mirror after τ_0 since that depends on the unknown future trajectory of the mirror. In the following, we shall propose a definition of moving-mirror entropy associated with the lack of information about the future trajectory of the mirror.

It is expected that this lack of information should be related to uncertainty about the quantum state of wave-packet modes which will be reflected by the mirror in the future since the latter is determined by the former, if the initial quantum state at $\mathcal{I}^-$ is fixed. In order to represent such uncertainty quantitatively, it is convenient to consider a quantum-mechanical generalization of conditional entropy, the so-called von Neumann conditional entropy.

Classically, the conditional entropy of an experiment A relative to another experiment B is defined by $H(A|B) = -\Sigma_{a,b} p(a,b) \ln p(a|b)$, where a and b represent outcomes of A and B, respectively, $p(a,b)$ is the joint probability of a and b, $p(a|b) = p(a,b)/p(b)$ is the conditional probability of a for a given outcome b, and $p(b)$ is the probability of b. The conditional entropy corresponds to uncertainty about the outcome of A after the experiment B has been performed. The quantum analogue of conditional entropy was considered in references [17,18]. Consider a Hilbert space $\mathcal{F}$ of the form $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ and let ρ be a density matrix on $\mathcal{F}$. Given ρ, the conditional entropy of the subsystem 1 relative to 2 is defined by

$$S_{1|2} = \mathbf{Tr}[\rho\sigma_{1|2}], \tag{1}$$

where

$$\sigma_{1|2} = \mathbf{1}_1 \otimes \ln \rho_2 - \ln \rho, \text{ and}$$

$$\rho_2 = \mathbf{Tr}_1 \rho. \tag{2}$$

In our case, $\mathcal{F}_1$ and $\mathcal{F}_2$ are symmetric Fock spaces constructed from Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of mode functions which are reflected by the mirror after and before τ_0, respectively. It seems natural to assume that the initial quantum state is the vacuum $|0,-\rangle$ determined by positive-frequency mode functions at $\mathcal{I}^-$. Correspondingly, the density matrix ρ represents a pure state given by $\rho = |0,-\rangle\langle 0,-|$. It is easy to see that, if ρ is pure,

$$S_{1|2} = -S_{ent}, \tag{3}$$

where S_{ent} is entanglement entropy defined by

$$S_{ent} = -\mathbf{Tr}_2[\rho_2 \ln \rho_2]. \tag{4}$$

Moreover, in this case, the quantum state $|0,-\rangle$ is the direct product of quantum states $|0,-R\rangle$ and $|0,-L\rangle$ in the right and left sectors R and L separated by the mirror. Hence, the entanglement entropy can be calculated separately in each of the two regions, and the total entanglement entropy is the sum of these two contributions.

To calculate the entanglement entropy in the R region, it is convenient to write the mirror trajectory as $v = p(u)$ and to introduce a new coordinate $\tilde{u}$ by $\tilde{u} \equiv p(u)$. In terms of this coordinate, the mirror trajectory becomes $v = \tilde{u}$ and the R region is $v \geqslant \tilde{u}$. Thence, as we shall briefly illustrate below, the entanglement entropy in the R region is easily calculated and given by

$$S_{entR} = \frac{1}{12} \ln[L_{\tilde{u}}/l_{\tilde{u}}], \tag{5}$$

where $L_{\tilde{u}}$ and $l_{\tilde{u}}$ are infrared and ultraviolet cutoffs in the $\tilde{u}$-coordinate [19].

Now let us verify this expression for the case of Dirichlet boundary conditions at the mirror. For this purpose note first that the entanglement entropy does not depend on the bases in $\mathcal{F}_1$ and $\mathcal{F}_2$ and so we can choose these bases in a convenient way. Hence, following Unruh [20], we introduce Rindler coordinates $\tilde{u}^{(1)}$ and $\tilde{u}^{(2)}$ defined by

$$\tilde{u}^{(1)} = -\ln(\tilde{u} - \tilde{u}_0) \quad \text{for} \quad \tilde{u} > \tilde{u}_0,$$

$$\tilde{u}^{(2)} = -\ln(-\tilde{u} + \tilde{u}_0) \quad \text{for} \quad \tilde{u} < \tilde{u}_0, \tag{6}$$

where $\tilde{u}_0 = p(X^u(\tau_0))$. Next, we can construct symmetric Fock spaces $\mathcal{F}_{1R}$ and $\mathcal{F}_{2R}$ from the following mode functions $\varphi_\omega^{(1)}$ and $\varphi_\omega^{(2)}$, respectively:

$$\varphi_\omega^{(1)} = \theta(\tilde{u} - \tilde{u}_0) e^{i\omega \tilde{u}^{(1)}} - \theta(v - \tilde{u}_0) e^{i\omega v^{(1)}},$$

$$\varphi_\omega^{(2)} = \theta(-\tilde{u} + \tilde{u}_0) e^{-i\omega \tilde{u}^{(2)}} - \theta(-v + \tilde{u}_0) e^{-i\omega v^{(2)}}, \tag{7}$$

where $v^{(1)}$ and $v^{(2)}$ are defined by

$$v^{(1)} = -\ln(v - \tilde{u}_0) \quad \text{for} \quad v > \tilde{u}_0,$$

$$v^{(2)} = -\ln(-v + \tilde{u}_0) \quad \text{for} \quad v < \tilde{u}_0. \tag{8}$$

Evidently, the Fock space $\mathcal{F}_{1R}$ (and $\mathcal{F}_{2R}$) is the subspace of $\mathcal{F}_1$ (and $\mathcal{F}_2$, respectively) which is relevant to the R region. Thirdly, the vacuum $|0,-R\rangle$ at $\mathcal{I}^-$ in the R region can be defined by the following set of mode functions $\Phi_\omega^{(1)}$ and $\Phi_\omega^{(2)}$ which are analytic in the lower complex v plane:

$$\Phi_\omega^{(1)} = N_\omega(\varphi_\omega^{(1)} + e^{-\pi\omega} \varphi_\omega^{(2)*}),$$

$$\Phi_\omega^{(2)} = N_\omega(\varphi_\omega^{(2)} + e^{-\pi\omega} \varphi_\omega^{(1)*}), \tag{9}$$

where N_ω is a normalization constant. Hence, following [21], it is now easy to expand the vacuum $|0,-R\rangle$ in terms of Fock space states constructed from $\{\varphi_\omega^{(1,2)}\}$ and to trace over the degrees of freedom in $\mathcal{F}_{1R}$. The resulting reduced density matrix on $\mathcal{F}_{2R}$ is the thermal density matrix with temperature $T = 1/2\pi$, provided that it is written in terms of Fock space states constructed from $\{\varphi_\omega^{(2)}\}$. Therefore the corresponding entropy density, or entropy per unit interval of the $\tilde{u}^{(2)}$ coordinate, is $\mathcal{S} = (\pi/6)T = 1/12$. Finally, by introducing the ultraviolet and infrared cutoffs $l_{\tilde{u}}$ and $L_{\tilde{u}}$, we obtain

$$S_{entR} = \int_{\tilde{u} = \tilde{u}_0 - L_{\tilde{u}}}^{\tilde{u} = \tilde{u}_0 - l_{\tilde{u}}} \mathcal{S} d\tilde{u}^{(2)} = \frac{1}{12} \ln[L_{\tilde{u}}/l_{\tilde{u}}]. \tag{10}$$

Similarly, to calculate the entanglement entropy in the L region, we write the mirror trajectory as $u = q(v)$ and introduce a new coordinate $\tilde{v}$ by $\tilde{v} \equiv q(v)$. Using this coordinate, the entanglement entropy in the L region is given by

$$S_{entL} = \frac{1}{12}\ln[L_{\tilde{v}}/l_{\tilde{v}}], \tag{11}$$

where $L_{\tilde{v}}$ and $l_{\tilde{v}}$ are infrared and ultraviolet cutoffs in the $\tilde{v}$-coordinate.

The ultraviolet cutoffs $l_{\tilde{u}}$ and $l_{\tilde{v}}$ can be related to the ultraviolet cutoff l in the proper time τ as follows:

$$l_{\tilde{u}} = \frac{d\tilde{u}}{d\tau}l + \frac{1}{2}\frac{d^2\tilde{u}}{d\tau^2}l^2 + \frac{1}{6}\frac{d^3\tilde{u}}{d\tau^3}l^3 + O(l^4),$$

$$l_{\tilde{v}} = \frac{d\tilde{v}}{d\tau}l + \frac{1}{2}\frac{d^2\tilde{v}}{d\tau^2}l^2 + \frac{1}{6}\frac{d^3\tilde{v}}{d\tau^3}l^3 + O(l^4), \tag{12}$$

or

$$l_{\tilde{u}} = (p'(u))^{1/2} l\left[1 + \frac{1}{2}la^\mu n_\mu + \frac{l^2}{6}\left(\frac{d}{d\tau}(a^\mu n_\mu) + a^2\right) + O(l^3a^3, l^3 a\partial_\tau a)\right],$$

$$l_{\tilde{v}} = (q'(v))^{1/2} l\left[1 - \frac{1}{2}la^\mu n_\mu + \frac{l^2}{6}\left(-\frac{d}{d\tau}(a^\mu n_\mu) + a^2\right) + O(l^3a^3, l^3 a\partial_\tau a)\right], \tag{13}$$

where $a^\mu = d^2X^\mu/d\tau^2$, $a^2 = a^\mu a_\mu$, and n^μ is the unit normal to the trajectory directed toward the R region. Since $p'q' = 1$, we obtain

$$l_{\tilde{u}}l_{\tilde{v}} = l^2\left[1 + \frac{1}{12}l^2a^2 + O(l^3a^3, l^3a\partial_\tau a)\right] \tag{14}$$

as a function of the proper time τ along the mirror trajectory. On the other hand, provided that the mirror is asymptotically inertial, we obtain $L_{\tilde{u}}L_{\tilde{v}} = L^2$, where L is the infrared cutoff in the proper time τ. Hence, we obtain the total entanglement entropy.

$$S_{ent} = S_{entR} + S_{entL} = -\frac{1}{144}l^2a^2[1 + O(la, l\partial_\tau \ln a)] + \frac{1}{6}\ln[L/l]. \tag{15}$$

Thus,

$$S_{1|2} = \frac{1}{144}l^2a^2[1 + O(la, l\partial_\tau \ln a)] - \frac{1}{6}\ln[L/l]. \tag{16}$$

Now we define *moving-mirror entropy* so that it measures how much the motion of the mirror increases the uncertainty of the quantum state of wave-packet modes which are reflected by the mirror after τ_0. The simplest procedure is to subtract $S_{1|2}$ for a non-accelerating mirror from $S_{1|2}$ for the actual trajectory:

$$S_{MM}(\tau_0) \equiv S_{1|2} - S_{1|2}|_{a=0} = \frac{1}{144}l^2a^2(\tau_0)[1 + O(la, l\partial_\tau \ln a)]. \tag{17}$$

In summary we have calculated von Neumann conditional entropy for a quantized scalar field in a two-dimensional Minkowski spacetime with a moving mirror and proposed a definition of moving-mirror entropy S_{MM} as the difference between von Neumann conditional entropy for the actual mirror trajectory and the inertial trajectory. We expect that S_{MM} is related to uncertainty about the future or temporarily inaccessible information. Since black-hole entropy has many interpretations [22–24], probably it is also worthwhile investigating other possible interpretations of the moving-mirror entropy.

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